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# Applications of Asymptotic Riesz Representation Theorem

Simona Macovei\*

**Abstract.** We review the relation between compact asymptotic spectral measures and certain positive asymptotic morphism on locally compact spaces via asymptotic Riesz representation theorem, as introduced by Martinez and Trout [3]. Applications to this theorem shall be discuss.

## 1 Introduction

In [3], Martinez and Trout introduced the positive asymptotic morphism, defined as:

A *positive asymptotic morphism* from a  $C^*$  - algebra  $A$  to a  $C^*$  - algebra  $B$  is a family of maps  $\{Q_h\}_{h \in (0,1]} : A \rightarrow B$ , parameterized by  $h \in (0,1]$ , such that the following hold:

1. Each  $Q_h$  is a positive linear map;
2. The map  $h \mapsto Q_h(f) : (0,1] \rightarrow B(H)$  is continuous for each  $f \in A$ ;
3. For all  $f, g \in A$  we have

$$\lim_{h \rightarrow 0} \|Q_h(fg) - Q_h(f)Q_h(g)\| = 0.$$

Also, Martinez and Trout introduced the concept of *an asymptotic spectral measure*  $\{A_h\}_{h \in (0,1]} : \Sigma \rightarrow B(H)$  associated to a measurable space  $(X, \Sigma)$  (Definition 3.1). Roughly, *an asymptotic spectral measure*  $\{A_h\}_{h \in (0,1]} : \Sigma \rightarrow B(H)$  is a continuous family of positive operator-valued measures which has the property:

$$\lim_{h \rightarrow 0} \|A_h(\Delta_1 \cap \Delta_2) - A_h(\Delta_1)A_h(\Delta_2)\| = 0,$$

for each  $\Delta_1, \Delta_2 \in \Sigma$ .

Let  $X$  be a locally compact Hausdorff topological space with Borel  $C^*$ -algebra  $\Sigma_X$  and let  $C_X \subset \Sigma_X$  denote the collection of all pre-compact open subsets of  $X$ . Let  $C_0(X)$  denote the  $C^*$  - algebra of all continuous functions which vanish at infinity on  $X$ . Define  $B_0(X)$  to be the  $C^*$  - subalgebra of  $B_b(X)$  ( $C^*$  - algebra of

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\*simonamacovei@yahoo.com

all bounded Borel functions on  $X$ ) generated by  $\{\chi_U | U \in C_X\}$ , where  $\chi_U$  denotes the characteristic function of  $U \subseteq X$ .

Let  $H$  be a separable Hilbert space,  $B(H)$  be the  $C^*$  - algebra of all bounded linear operators on  $H$  and  $B$  denote a hereditary  $C^*$  - algebra of  $B(H)$ .

The asymptotic Riesz representation theorem, formulated by Martinez and Trout [3], gives a bijective correspondence between positive asymptotic morphisms  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$ , having property  $Q_h(C_0(X)) \subset B$  for any  $h \in (0,1]$ , and asymptotic spectral measures  $\{A_h\}_{h \in (0,1]} : X \rightarrow B(H)$ , having property  $A_h(C_X) \subset B$  for any  $h \in (0,1]$ . This correspondence is given by

$$Q_h(f) = \int_X f(x) dA_h(x),$$

for any  $f \in B_0(X)$ . (Teorema 4.2. [3])

In this paper, we study some applications of this theorem.

## 2 Positive Asymptotic Morphisms

In this section we review the basic definitions and properties of positive asymptotic morphisms.

Let  $A$  and  $B$  be two  $C^*$  - algebras. A linear operator  $Q : A \rightarrow B$  is *positive* if  $Q(f) \geq 0$  for any  $f \geq 0$ .

**Definition 1.** A *positive asymptotic morphism* from a  $C^*$  - algebra  $A$  to a  $C^*$  - algebra  $B$  is a family of maps  $\{Q_h\}_{h \in (0,1]} : A \rightarrow B$ , parameterized by  $h \in (0,1]$ , such that the following hold:

1. Each  $Q_h$  is a positive linear map;
2. The map  $h \mapsto Q_h(f) : (0,1] \rightarrow B(H)$  is continuous for each  $f \in A$ ;
3. For all  $f, g \in A$  we have

$$\lim_{h \rightarrow 0} \|Q_h(fg) - Q_h(f)Q_h(g)\| = 0.$$

(Definition 2.1. [3])

**Definition 2.** Two positive asymptotic morphisms  $\{Q_h\}, \{P_h\}_{h \in (0,1]} : A \rightarrow B$  are called *asymptotically equivalent* if for all  $f \in A$  we have that

$$\lim_{h \rightarrow 0} \|Q_h(f) - P_h(f)\| = 0.$$

(Definition 2.2. [3])

**Remark 3.** *The asymptotic equivalence relation of two positive asymptotic morphisms is symmetric, reflexive and transitive.*

Let  $H$  be a separable Hilbert space and  $B(H)$  be the  $C^*$ -algebra of all bounded linear operators on  $H$ . Let  $X$  be a set equipped with a  $C^*$ -algebra  $\Sigma_X$  of measurable sets and let  $C_X \subset \Sigma_X$  denote the collection of all pre-compact open subsets of  $X$ . Define  $B_0(X)$  to be the  $C^*$ -subalgebra of  $B_b(X)$  ( $C^*$ -algebra of all bounded Borel functions on  $X$ ) generated by  $\{\chi_U | U \in C_X\}$ , where  $\chi_U$  denotes the characteristic function of  $U \subseteq X$ . If  $X$  is also  $\sigma$ -compact, then let  $C_0(X)$  be the set of all continuous functions which vanish at infinity on  $X$ .

We call *support* of a morphism  $Q : C_0(X) \rightarrow B(H)$  the set

$$\text{supp}(Q) = \cap \{F \subset X | F \text{ closed and } Q(f) = 0, \forall f \text{ cu } \text{supp}(f) \subset X \setminus F\}.$$

A morphism  $Q : C_0(X) \rightarrow B(H)$  will be said to have *compact support* if there is a compact subset  $K$  of  $X$  such that  $\text{supp}(Q) \subset K$ .

**Definition 4.** *A positive asymptotic morphism  $\{Q_h\}_{h \in (0,1]} : B_b(X) \rightarrow B(H)$  will be said to have compact support if there is a compact subset  $K$  of  $X$  such that  $\text{supp}(Q_h) \subset K, \forall h \in (0,1]$ .*

**Definition 5.** *Let  $\{Q_h\}_{h \in (0,1]} : B_b(X) \rightarrow B(H)$  be a positive asymptotic morphism. The support of  $\{Q_h\}$  is defined as the set*

$$\text{supp}(\{Q_h\}) = \cap \left\{ F \text{ closed} \mid \lim_{h \rightarrow 0} \|Q_h(f)\| = 0, \forall f \in B_b(X) \text{ with } \text{supp}(f) \cap F = \emptyset \right\}.$$

**Remark 6.** 1.  $\text{supp}(\{Q_h\}) \subseteq \bigcup_{h \in (0,1]} \text{supp}(Q_h)$ .

2. If  $\{Q_h\}$  has compact support  $\text{supp}(\{Q_h\})$  is a compact set.

**Theorem 7.** *Let  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  be a positive asymptotic morphism such that  $\lim_{h \rightarrow 0} \|Q_h(1) - I\| = 0$ . Then*

$$Sp(\{Q_h(f)\}) \subseteq f(\text{supp}(\{Q_h\})),$$

for any  $f \in B_0(X)$ .

*Proof.* Let  $f \in B_0(X)$  and  $\lambda \notin f(\text{supp}(\{Q_h\}))$ . Then  $\lambda - f(x) \neq 0, \forall x \in \text{supp}(\{Q_h\})$ . Thus there is an open set  $G \supset \text{supp}(\{Q_h\})$  such that  $\lambda - f(x) \neq 0, \forall x \in G$ . Therefore, the application  $x \mapsto 1/(f(x) - \lambda) \in C(G)$ .

Let  $g \in B_0(X)$  such that  $g(x) = 1/(f(x) - \lambda), \forall x \in G$ . Then

$$g(x)(f(x) - \lambda) = (f(x) - \lambda)g(x) = 1,$$

$\forall x \in G$ . Taking into account the above relation and since  $\lim_{h \rightarrow 0} \|Q_h(1) - I\| = 0$ , it follows that

$$\begin{aligned} & \lim_{h \rightarrow 0} \|Q_h(g)(\lambda - Q_h(f)) - I\| = \\ & \lim_{h \rightarrow 0} \|Q_h(g)(\lambda - Q_h(f)) - Q_h(g)Q_h(\lambda - f) + Q_h(g)Q_h(\lambda - f) - Q_h(g(\lambda - f)) + \\ & \quad Q_h(g(\lambda - f)) - Q_h(1) + Q_h(1) - I\| \leq \\ & \leq \lim_{h \rightarrow 0} \|Q_h(g)(\lambda - Q_h(f)) - Q_h(g)Q_h(\lambda - f)\| + \\ & \quad + \lim_{h \rightarrow 0} \|Q_h(g)Q_h(\lambda - f) - Q_h(g(\lambda - f))\| + \\ & \quad + \lim_{h \rightarrow 0} \|Q_h(g(\lambda - f)) - Q_h(1)\| + \lim_{h \rightarrow 0} \|Q_h(1) - I\| = 0. \end{aligned}$$

Analogously  $\lim_{h \rightarrow 0} \|(\lambda - Q_h(f))Q_h(g) - I\| = 0$ . Therefore  $\lambda \in r(\{Q_h(f)\})$ .

We have showed that

$$Sp(\{Q_h(f)\}) \subseteq f(supp(\{Q_h\})), \forall f \in B_0(X).$$

□

**Corollary 8.** *Let  $\{Q_h\}_{h \in (0,1]} : B_0(\mathbb{C}) \rightarrow B(H)$  be a positive asymptotic morphism such that  $\lim_{h \rightarrow 0} \|Q_h(1) - I\| = 0$ . Then*

$$Sp(\{Q_h(z)\}) \subseteq supp(\{Q_h\}).$$

*Proof.* We take in Theorem 7  $f = z$ , where  $z$  represents the identity application.

□

### 3 Asymptotic Spectral Measures

Let  $(X, \Sigma)$  be a measurable space and  $H$  a separable Hilbert space. Let  $\varepsilon \subset \Sigma$  be a fix collection of measurable sets.

**Definition 9.** *A positive operator-valued measure on the measurable space  $(X, \Sigma)$  is a mapping  $A : \Sigma \rightarrow B(H)$  which satisfies the following properties:*

1.  $A(\emptyset) = 0$ ;
2.  $A(\Delta) \geq 0, \forall \Delta \in \Sigma$ ;
3.  $A(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} A(\Delta_n)$ , for disjoint measurable sets  $(\Delta_n)_{n=1}^{\infty} \subset \Sigma$ , where the series converges in weak operator topology.

**Definition 10.** An asymptotic spectral measure on  $(X, \Sigma, \varepsilon)$  is a family of maps  $\{A_h\}_{h \in (0,1]} : \Sigma \rightarrow B(H)$ , parameterized by  $h \in (0,1]$ , such that the following hold:

- i) Each  $A_h$  is a positive operator-valued measure;
- ii)  $\lim_{h \rightarrow 0} \|A_h(X)\| \leq 1$  ;
- iii) The map  $h \mapsto A_h(\Delta) : (0,1] \rightarrow B(H)$  is continuous, for any  $\Delta \in \varepsilon$ ;
- iv) For each  $\Delta_1, \Delta_2 \in \varepsilon$  we have

$$\lim_{h \rightarrow 0} \|A_h(\Delta_1 \cap \Delta_2) - A_h(\Delta_1)A_h(\Delta_2)\| = 0.$$

The triple  $(X, \Sigma, \varepsilon)$  will be called asymptotic measure space.

If  $\varepsilon = \Sigma$ , then  $\{A_h\}$  will be called total (full) asymptotic spectral measure on  $(X, \Sigma)$ .

If each  $A_h$  is normalized, i.e.  $A_h(X) = I_H$ , then  $\{A_h\}$  will be called normalized. (Definition 3.1. [3])

If  $\lim_{h \rightarrow 0} \|A_h(X) - I_H\| = 0$ , then  $\{A_h\}$  will be called asymptotically normalized.

**Definition 11.** Two asymptotic spectral measures  $\{A_h\}, \{B_h\}_{h \in (0,1]} : \Sigma \rightarrow B(H)$  on  $(X, \Sigma)$  are said to be (asymptotically) equivalent if for each measurable set  $\Delta \in \varepsilon$ , we have

$$\lim_{h \rightarrow 0} \|A_h(\Delta) - B_h(\Delta)\| = 0.$$

(Definition 2.2 [3])

Let  $X$  denote a locally compact Hausdorff topological space with Borel  $\sigma$ -algebra  $\Sigma$ .

**Definition 12.** Let  $A$  be a Borel positive operator-valued measure on  $X$ . The cospectrum of  $A$  is defined as the set

$$\text{cospec}(A) = \bigcup \{U \subset X \mid U \text{ is open and } A(U) = 0\}.$$

The spectrum of  $A$  is the complement, i.e.

$$\text{spec}(A) = X \setminus \text{cospec}(A).$$

**Definition 13.** Let  $A$  be a Borel positive operator-valued measure on  $X$ .  $A$  is said to be compact if  $\text{spec}(A)$  is a compact subset of  $X$ .

**Theorem 14.** Let  $A$  be a Borel positive operator-valued measure on  $X$ . Then

$$A(\text{spec}(A)) = A(X).$$

(Theorem 23 [5])

**Definition 15.** An asymptotic spectral measure  $\{A_h\}$  on  $X$  will have compact support if there is a compact subset  $K$  of  $X$  such that  $\text{spec}(A_h) \subset K, \forall h \in (0, 1]$ . (Definition 3.4 [3])

**Remark 16.** i) If  $\{A_h\}$  has compact, then  $A_h$  has compact support,  $\forall h \in (0, 1]$ ;  
ii) If  $\{A_h\}$  has compact support, then

$$A_h(\text{spec}(A_h)) = A_h(K) = A_h(X), \forall h \in (0, 1].$$

**Definition 17.** Let  $\{A_h\}_{h \in (0, 1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure. The cospectrum of  $\{A_h\}$  is defined as the set

$$\text{cospec}(\{A_h\}) = \bigcup \left\{ a \subset X \mid a \text{ open and } \lim_{h \rightarrow 0} \|A_h(a)\| = 0 \right\}.$$

The spectrum of  $\{A_h\}$  is the complement of  $\text{cospec}(\{A_h\})$ , i.e.

$$\text{spec}(\{A_h\}) = X \setminus \text{cospec}(\{A_h\}).$$

**Remark 18.** i)  $\text{spec}(\{A_h\}) \subseteq \bigcup_{h \in (0, 1]} \text{spec}(A_h)$  and  $\bigcap_{h \in (0, 1]} \text{cospec}(A_h) \subseteq \text{cospec}(\{A_h\})$ .  
ii) If  $\{A_h\}$  has compact support, then  $\text{spec}(\{A_h\})$  is also a compact set.

*Proof.* i) Let  $a \subset \bigcap_{h \in (0, 1]} \text{cospec}(A_h)$  be an open set. Thus  $A_h(a) = 0, \forall h \in (0, 1]$  and

$$\lim_{h \rightarrow 0} \|A_h(a)\| = 0.$$

Therefore

$$a \subset \text{cospec}(\{A_h\}), \forall a \subset \bigcap_{h \in (0, 1]} \text{cospec}(A_h).$$

It results

$$\bigcap_{h \in (0, 1]} \text{cospec}(A_h) \subseteq \text{cospec}(\{A_h\})$$

and, taking the complement we have

$$\text{spec}(\{A_h\}) \subseteq \bigcup_{h \in (0, 1]} \text{spec}(A_h).$$

ii) Let  $\{A_h\}$  be a compact set. Thus there is a compact subset  $K$  of  $X$  such that  $\text{spec}(A_h) \subset K, \forall h \in (0, 1]$ .

By i), it follows

$$\text{spec}(\{A_h\}) \subseteq \bigcup_{h \in (0,1]} \text{spec}(A_h) \subset K.$$

□

**Lemma 19.** *Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure. Then*

$$\lim_{h \rightarrow 0} \|A_h(K)\| = 0,$$

for each compact subset  $K$  of  $\text{cospec}(\{A_h\})$ .

*Proof.* Let  $K$  be a compact subset of  $\text{cospec}(\{A_h\})$ . Thus each element of  $K$  belongs to an open set  $a$  having property  $\lim_{h \rightarrow 0} \|A_h(a)\| = 0$ . Since  $K$  is a compact set, hence there is a family of open set  $(a_i)_1^n \subset X$  such that  $K \subset a_1 \cup \dots \cup a_n$ . Therefore

$$A(K) \leq A(a_1) + \dots + A(a_n) = 0$$

and

$$\lim_{h \rightarrow 0} \|A_h(K)\| \leq \lim_{h \rightarrow 0} \|A_h(a_1)\| + \dots + \lim_{h \rightarrow 0} \|A_h(a_n)\| = 0$$

□

**Proposition 20.** *Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure. Then*

$$\overline{\lim}_{h \rightarrow 0} \|A_h(X)\| = \overline{\lim}_{h \rightarrow 0} \|A_h(\text{spec}(\{A_h\}))\|.$$

*Proof.* We show that

$$\lim_{h \rightarrow 0} \|A_h(\text{cospec}(\{A_h\}))\| = 0.$$

Let  $K$  be a compact subset of  $\text{cospec}(\{A_h\})$ . By Lemma 19, it follows that

$$\lim_{h \rightarrow 0} \|A_h(K)\| = 0.$$

Since  $A_h$  is regular, for any  $h \in (0,1]$ , by above relation, it results that

$$\lim_{h \rightarrow 0} \|A_h(\text{cospec}(\{A_h\}))\| = 0.$$

Since

$$\text{spec}(\{A_h\}) = X \setminus \text{cospec}(\{A_h\}),$$

we have that

$$\overline{\lim}_{h \rightarrow 0} \|A_h(X)\| = \overline{\lim}_{h \rightarrow 0} \|A_h(\text{spec}(\{A_h\})) + A_h(\text{cospec}(\{A_h\}))\| \leq$$

$$\begin{aligned} &\leq \overline{\lim}_{h \rightarrow 0} \|A_h(\text{spec}(\{A_h\}))\| + \overline{\lim}_{h \rightarrow 0} \|A_h(\text{cospec}(\{A_h\}))\| = \\ &= \overline{\lim}_{h \rightarrow 0} \|A_h(\text{spec}(\{A_h\}))\|. \end{aligned}$$

In addition, we have that

$$\begin{aligned} &\overline{\lim}_{h \rightarrow 0} \|A_h(\text{spec}(\{A_h\}))\| = \\ &= \overline{\lim}_{h \rightarrow 0} \|A_h(\text{spec}(\{A_h\})) + A_h(\text{cospec}(\{A_h\})) - A_h(\text{cospec}(\{A_h\}))\| \leq \\ &\leq \overline{\lim}_{h \rightarrow 0} \|A_h(\text{cospec}(\{A_h\}))\| + \overline{\lim}_{h \rightarrow 0} \|A_h(\text{cospec}(\{A_h\})) + A_h(\text{spec}(\{A_h\}))\| \leq \\ &\leq \overline{\lim}_{h \rightarrow 0} \|A_h(X)\|. \end{aligned}$$

From two preceding relations, it follows that

$$\overline{\lim}_{h \rightarrow 0} \|A_h(X)\| = \overline{\lim}_{h \rightarrow 0} \|A_h(\text{spec}(\{A_h\}))\|.$$

□

**Theorem 21.** *Let  $\{A_h\}, \{B_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be two asymptotic spectral measures. If  $\{A_h\}, \{B_h\}$  are asymptotically equivalent, then*

$$\text{spec}(\{A_h\}) = \text{spec}(\{B_h\}).$$

*Proof.* Let be an open set  $a \subset \text{cospec}(\{A_h\})$ . Thus

$$\lim_{h \rightarrow 0} \|A_h(a)\| = 0.$$

Since  $\{A_h\}, \{B_h\}$  are asymptotically equivalent, it results that

$$\lim_{h \rightarrow 0} \|A_h(a) - B_h(a)\| = 0.$$

By two preceding relations, we have that

$$\lim_{h \rightarrow 0} \|B_h(a)\| = \lim_{h \rightarrow 0} \|A_h(a)\| = 0.$$

Thus  $a \subset \text{cospec}(\{B_h\}), \forall a \subset \text{cospec}(\{A_h\})$  open. Therefore

$$\text{spec}(\{B_h\}) \subset \text{spec}(\{A_h\}).$$

Reciprocal: Analog.

□

**Remark 22.** *Let  $\{A_h\}, \{B_h\}$  be two asymptotic spectral measures on  $(X, \Sigma)$ . If  $\{A_h\}, \{B_h\}$  are asymptotically equivalent, then  $\{A_h\}$  has compact support if and only if  $\{B_h\}$  has compact support.*



*Proof.* By preceding Proposition, for each compact subset  $K$  of  $X$ , we have  $\text{spec}(\{A_h\}) \subset K$  if and only if  $\text{spec}(\{B_h\}) \subset K$ . □

**Proposition 23.** *Let  $\{A_h\}$  be a full asymptotic spectral measure on  $(X, B)$ , where  $B$  is the  $\sigma$  - algebra of Borel subsets of  $X$ , and  $a \in B$ . Then  $\{A_h^a\} : B \rightarrow B(H)$ , parameterized by  $h \in (0, 1]$ , given by*

$$A_h^a(b) = A_h(a \cap b), \forall b \in B \text{ and } \forall h \in (0, 1],$$

*is an asymptotic spectral measure.*

*Proof.* By definition of  $\{A_h^a\}$ , we have

$$A_h^a(\emptyset) = A_h(a \cap \emptyset) = A_h(\emptyset) = 0, \forall h \in (0, 1]$$

and

$$\begin{aligned} \overline{\lim}_{h \rightarrow 0} \|A_h^a(X)\| &= \overline{\lim}_{h \rightarrow 0} \|A_h(a \cap X)\| = \\ &= \overline{\lim}_{h \rightarrow 0} \|A_h(a)\| \leq \overline{\lim}_{h \rightarrow 0} \|A_h(X)\| \leq 1. \end{aligned}$$

Let  $(b_n)_{n \in \mathbb{N}} \subset B$  be a family of disjoint sets. Thus  $(a \cap b_n)_{n \in \mathbb{N}} \subset B$  is also a family of disjoint sets. Since  $A_h$  is numerable additive,  $\forall h \in (0, 1]$ , it results

$$\begin{aligned} A_h^a\left(\bigcup_{n \in \mathbb{N}} b_n\right) &= A_h\left(\bigcup_{n \in \mathbb{N}} (a \cap b_n)\right) = \\ &= \sum_{n \in \mathbb{N}} A_h(a \cap b_n) = \sum_{n \in \mathbb{N}} A_h^a(b_n), \forall h \in (0, 1]. \end{aligned}$$

As the map  $(0, 1] \rightarrow B(H) : h \rightarrow A_h(a \cap b)$  is continuous  $\forall b \in B$ , then the map is also continuous  $\forall b \in B$ .

Let  $b_1, b_2 \in B$ . Thus

$$\begin{aligned} \overline{\lim}_{h \rightarrow 0} \left\| A_h^a(b_1 \cap b_2) - A_h^a(b_1)A_h^a(b_2) \right\| &= \\ &= \overline{\lim}_{h \rightarrow 0} \left\| A_h(a \cap b_1 \cap b_2) - A_h(a \cap b_1)A_h(a \cap b_2) \right\| = \\ &= \lim_{h \rightarrow 0} \left\| A_h((a \cap b_1) \cap (a \cap b_2)) - A_h(a \cap b_1)A_h(a \cap b_2) \right\| = 0. \end{aligned}$$

Therefore,  $\{A_h^a\} : B \rightarrow B(H)$  is a full asymptotic spectral measure. □

**Proposition 24.** *Let  $\{A_h\}$  be a full asymptotic spectral measure on  $(X, B)$  and  $\{A_h^a\} : B \rightarrow B(H)$ , parameterized by  $h \in (0, 1]$ , given by  $A_h^a(b) = A_h(a \cap b)$ ,  $\forall b \in B$  and  $\forall h \in (0, 1]$ . Then*

$$\text{spec}(A_h^a) \subseteq \bar{a} \cap \text{spec}(A_h), \forall h \in (0, 1].$$

*Proof.* Let  $b$  be a compact set such that  $b \subset \mathbb{C} \setminus \bar{a}$ . Thus  $a \cap b = \emptyset$ . By this relation we have

$$A_h^a(b) = A_h(a \cap b) = A_h(\emptyset) = 0, \forall h \in (0, 1],$$

hence

$$b \subset \text{cospec}(A_h^a) \Rightarrow \mathbb{C} \setminus \bar{a} \subset \text{cospec}(A_h^a), \forall h \in (0, 1]$$

(by regularity property of measures  $A_h$  - Teorema 23 [5]). Therefore

$$\text{spec}(A_h^a) \subseteq \bar{a}, \forall h \in (0, 1].$$

Let  $b$  be a compact set such that  $b \subset \mathbb{C} \setminus \text{spec}(A_h)$ . Thus there is a family of open sets  $(b_i)_{i=1, n}$  such that

$$b \subset \bigcup_{i=1}^n b_i, \quad b_i \subset \mathbb{C} \setminus \text{spec}(A_h) \Rightarrow A_h(b_i) = 0.$$

Since each  $A_h$  is additive, we have

$$A_h(b) \leq A_h\left(\bigcup_{i=1}^n b_i\right) = \sum_{i=1}^n A_h(b_i) = 0.$$

Taking into account the following relation

$$A_h^a(b) = A_h(a \cap b) \leq \sum_{i=1}^n A_h(a \cap b_i) \leq \sum_{i=1}^n A_h(b_i) = 0$$

it results

$$b \subset \mathbb{C} \setminus \text{spec}(A_h^a),$$

for any compact set  $b$  such that  $b \subset \mathbb{C} \setminus \text{spec}(A_h)$ . Since each  $A_h$  is regular, it follows

$$\mathbb{C} \setminus \text{spec}(A_h) \subseteq \mathbb{C} \setminus \text{spec}(A_h^a), \forall h \in (0, 1].$$

Therefore

$$\text{spec}(A_h^a) \subseteq \text{spec}(A_h), \forall h \in (0, 1].$$

□

**Remark 25.** If  $\{A_h\}$  is an asymptotic spectral measure having compact support, then  $\{A_h^a\}$  is an asymptotic spectral measure having compact support,  $\forall a \in B$ .

**Proposition 26.** Two full asymptotic spectral measures on  $(X, B)$   $\{A_h\}, \{B_h\}$  are asymptotically equivalent if and only if  $\{A_h^a\}, \{B_h^a\} : B \rightarrow B(H)$ , given by  $A_h^a(b) = A_h(a \cap b)$  and  $B_h^a(b) = B_h(a \cap b)$ ,  $\forall b \in B, \forall h \in (0, 1]$ , are asymptotically equivalent  $\forall a \in B$ .

*Proof.* Let  $a \in B$  be fixed. Since  $\{A_h\}, \{B_h\}$  are asymptotically equivalent, thus

$$\lim_{h \rightarrow 0} \|A_h(a \cap b) - B_h(a \cap b)\| = 0, \forall b \in B.$$

It follows that

$$\lim_{h \rightarrow 0} \|A_h^a(b) - B_h^a(b)\| = 0, \forall b \in B.$$

Reciprocal. Since  $\{A_h^a\}, \{B_h^a\}$  are asymptotically equivalent  $\forall a \in B$  and

$$A_h^a(a) = A_h(a), B_h^a(a) = B_h(a),$$

it results

$$\begin{aligned} & \lim_{h \rightarrow 0} \|A_h(a) - B_h(a)\| = \\ & = \lim_{h \rightarrow 0} \|A_h^a(a) - B_h^a(a)\| = 0, \forall a \in B. \end{aligned}$$

Therefore,  $\{A_h\}, \{B_h\}$  are asymptotically equivalent. □

## 4 Asymptotic Riesz Representation Theorem

Let  $X$  be a locally compact Hausdorff topological space with Borel  $\sigma$ -algebra  $\Sigma_X$  and let  $C_X \subset \Sigma_X$  denote the collection of all pre-compact open subsets of  $X$ .

Let  $H$  be a separable Hilbert space,  $B(H)$  be the  $C^*$ -algebra of all bounded linear operators on  $H$  and  $B$  denote a hereditary  $C^*$ -algebra of  $B(H)$ .

**Lemma 27.** There is a bijective correspondence between Borel positive operator-valued measure  $A: \Sigma_X \rightarrow B(H)$ , having property  $A(C_X) \subset B$ , and positive morphism  $Q: C_0(X) \rightarrow B$ . This correspondence is given by

$$Q(f) = \int_X f(x) dA(x).$$

(Lemma 4.1. [3])

Let  $C_0(X)$  denote the  $C^*$  - algebra of all continuous functions which vanish at infinity on  $X$ . Define  $B_0(X)$  to be the  $C^*$  - subalgebra of  $B_b(X)$  ( $C^*$  - algebra of all bounded Borel functions on  $X$ ) generated by  $\{\chi_U | U \in C_X\}$ , where  $\chi_U$  denotes the characteristic function of  $U \subseteq X$ .

**Proposition 28.** *If  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  is a compact asymptotic spectral measure, then  $\{A_h\}$  verifies property  $A_h(C_X) \subset B$ ,  $\forall h \in (0,1]$ , where  $B$  is the hereditary subalgebra generated by  $\{A_h(\text{spec}(A_h))\}_{h \in (0,1]}$ .*

*Proof.* By Theorem 14 we have

$$A_h(\text{cospec}(A_h)) = 0.$$

Let  $a \in C_X$ . Then

$$0 \leq A_h(a \cap \text{cospec}(A_h)) \leq A_h(\text{cospec}(A_h)) = 0$$

and thus

$$0 \leq A_h(a \cap \text{spec}(A_h)) \leq A_h(\text{spec}(A_h)).$$

Since  $B$  is the hereditary subalgebra generated by  $\{A_h(\text{spec}(A_h))\}_{h \in (0,1]}$ , then  $A_h(a) \in B$ .

□

**Theorem 29.** (*Asymptotic Riesz Representation Theorem*): *There is a bijective correspondence between positive asymptotic morphisms  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$ , having property  $Q_h(C_0(X)) \subset B$ ,  $\forall h \in (0,1]$ , and asymptotic spectral measures  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$ , having property  $A_h(C_X) \subset B$ ,  $\forall h \in (0,1]$ , given by*

$$Q_h(f) = \int_X f(x) dA_h(x), \quad \forall f \in B_0(X).$$

(Theorem 4.2. [3])

## 5 Application of Asymptotic Riesz Representation Theorem

**Proposition 30.** *Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure, as in asymptotic Riesz representation theorem, and  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  the corresponding positive asymptotic morphism. Then the following assertions hold:*

1.  $\{Q_h\}$  is unitary if and only if  $\{A_h\}$  is normalized;

2.  $\lim_{h \rightarrow 0} \|A_h(X) - I_H\| = 0$  if and only if  $\lim_{h \rightarrow 0} \|Q_h(1) - I_H\| = 0$ ;
3. Let  $\{T_h\} \subset B(H)$ . Then  $\lim_{h \rightarrow 0} \|T_h Q_h(f) - Q_h(f) T_h\| = 0, \forall f \in B_0(X)$  if and only if  $\lim_{h \rightarrow 0} \|T_h A_h(\Delta) - A_h(\Delta) T_h\| = 0, \forall \Delta \in \Sigma_X$ .

*Proof.* i)  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  is unitary if  $Q_h(1) = I_H, \forall h \in (0,1]$ . Since

$$A_h(X) = Q_h(\chi_X) = \int_X \chi_X(x) dA_h(x) = \int_X dA_h(x) = Q_h(1) = I_H, \forall h \in (0,1],$$

it follows that  $\{A_h\}_{h \in (0,1]}$  is normalized.

Reciprocal. Since  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  is normalized, i.e.  $A_h(X) = I_H, \forall h \in (0,1]$ , then taking  $f = 1$  we have

$$Q_h(1) = \int_X dA_h(x) = A_h(X) = I_H, \forall h \in (0,1].$$

ii) It results from  $A_h(X) = Q_h(1), \forall h \in (0,1]$ .

iii) Since

$$\lim_{h \rightarrow 0} \|T_h Q_h(f) - Q_h(f) T_h\| = 0, \forall f \in S_0(X) \subset B_0(X),$$

taking  $f = \chi_\Delta$  it follows

$$\lim_{h \rightarrow 0} \|T_h A_h(\Delta) - A_h(\Delta) T_h\| = 0, \forall \Delta \in \Sigma_X.$$

Reciprocal. Since

$$\lim_{h \rightarrow 0} \|T_h A_h(\Delta) - A_h(\Delta) T_h\| = 0, \forall \Delta \in \Sigma_X,$$

and having in view  $A_h(\Delta) = Q_h(\chi_\Delta)$  it results

$$\lim_{h \rightarrow 0} \|T_h Q_h(\chi_\Delta) - Q_h(\chi_\Delta) T_h\| = 0, \forall \Delta \in \Sigma_X.$$

Let  $f \in S_0(X)$ . Then there are disjoint sets  $(\Delta_i)_{i=1,n}$  such that  $f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}$ . By above relation, we have

$$\begin{aligned} \overline{\lim}_{h \rightarrow 0} \|T_h Q_h(f) - Q_h(f) T_h\| &= \overline{\lim}_{h \rightarrow 0} \left\| \sum_{i=1}^n \alpha_i (T_h Q_h(\chi_{\Delta_i}) - Q_h(\chi_{\Delta_i}) T_h) \right\| \leq \\ &\leq \sum_{i=1}^n \overline{\lim}_{h \rightarrow 0} |\alpha_i| \|T_h Q_h(\chi_{\Delta_i}) - Q_h(\chi_{\Delta_i}) T_h\| \leq \end{aligned}$$

$$\leq \sum_{i=1}^n |\alpha_i| \overline{\lim}_{h \rightarrow 0} \|T_h Q_h(\chi_{\Delta_i}) - Q_h(\chi_{\Delta_i}) T_h\| = 0.$$

Let  $f \in B_0(X)$ . Then there are functions  $(f_n)_{n \in \mathbb{N}} \subset S_0(X)$  such that  $f \rightarrow f_n$ . By preceding relation, we have

$$\lim_{h \rightarrow 0} \|T_h Q_h(f) - Q_h(f) T_h\| = 0, \forall f \in B_0(X).$$

□

**Proposition 31.** *Two asymptotic spectral measures having compact support  $\{A_h\}, \{B_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  are asymptotically commutative, i.e.  $\lim_{h \rightarrow 0} \|A_h(\Delta_1) B_h(\Delta_2) - B_h(\Delta_2) A_h(\Delta_1)\| = 0, \forall \Delta_1, \Delta_2 \in \Sigma_X$ , if and only if the corresponding positive asymptotic morphisms  $\{Q_h\}, \{P_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  are asymptotically commutative, i.e.  $\lim_{h \rightarrow 0} \|Q_h(f) P_h(g) - P_h(g) Q_h(f)\| = 0, \forall f, g \in B_0(X)$ .*

*Proof.* Since  $A_h(\Delta_1) = Q_h(\chi_{\Delta_1}) \in B(H)$  and taking into account Proposition 30 iii), we have

$$\lim_{h \rightarrow 0} \|Q_h(\chi_{\Delta_1}) P_h(g) - P_h(g) Q_h(\chi_{\Delta_1})\| = 0,$$

$\forall \Delta_1 \in \Sigma_X$  and  $\forall g \in B_0(X)$ .

Let  $f \in S_0(X)$ . Then there are disjoint sets  $(\Delta_i)_{i=1,n}$  such that  $f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}$ . By above relation, we have

$$\begin{aligned} & \overline{\lim}_{h \rightarrow 0} \|Q_h(f) P_h(g) - P_h(g) Q_h(f)\| = \\ &= \overline{\lim}_{h \rightarrow 0} \left\| \sum_{i=1}^n \alpha_i (Q_h(\chi_{\Delta_i}) P_h(g) - P_h(g) Q_h(\chi_{\Delta_i})) \right\| \leq \\ &\leq \sum_{i=1}^n \overline{\lim}_{h \rightarrow 0} |\alpha_i| \|Q_h(\chi_{\Delta_i}) P_h(g) - P_h(g) Q_h(\chi_{\Delta_i})\| = \\ &= \sum_{i=1}^n |\alpha_i| \lim_{h \rightarrow 0} \|Q_h(\chi_{\Delta_i}) P_h(g) - P_h(g) Q_h(\chi_{\Delta_i})\| = 0. \end{aligned}$$

Let  $f \in B_0(X)$ . Then there are functions  $(f_n)_{n \in \mathbb{N}} \subset S_0(X)$  such that  $f \rightarrow f_n$ . By preceding relation, we have

$$\lim_{h \rightarrow 0} \|Q_h(f) P_h(g) - P_h(g) Q_h(f)\| = 0, \forall f, g \in B_b(X).$$

Reciprocal. Since

$$\lim_{h \rightarrow 0} \|Q_h(f) P_h(g) - P_h(g) Q_h(f)\| = 0, \quad f, g \in S_0(X) \subset B_0(X),$$

taking  $f = \chi_{\Delta_1}$  and  $g = \chi_{\Delta_2}$  follows

$$\lim_{h \rightarrow 0} \|A_h(\Delta_1) B_h(\Delta_2) - B_h(\Delta_2) A_h(\Delta_1)\| = 0, \quad \forall \Delta_1, \Delta_2 \in \Sigma_X.$$

□

**Theorem 32.** *Two asymptotic spectral measures having compact support are asymptotically equivalent if and only if the corresponding positive asymptotic morphisms are asymptotically equivalent.*

*Proof.* Let  $\{A_h\}, \{B_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be two asymptotic spectral measures having compact support and  $\{Q_h\}, \{P_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  the corresponding positive asymptotic morphisms. We suppose that  $\{A_h\}, \{B_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  are asymptotically equivalent, i.e.

$$\lim_{h \rightarrow 0} \|A_h(\Delta) - B_h(\Delta)\| = 0, \quad \forall \Delta \in \Sigma_X.$$

Since

$$A_h(\Delta) = Q_h(\chi_\Delta), B_h(\Delta) = P_h(\chi_\Delta),$$

from preceding relation, we obtain

$$\lim_{h \rightarrow 0} \|Q_h(\chi_\Delta) - P_h(\chi_\Delta)\| = 0, \quad \forall \Delta \in \Sigma_X.$$

Let  $f \in S_0(X)$ . Then there are disjoint sets  $(\Delta_i)_{i=1,n}$  such that  $f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}$ . By above relation, we have

$$\begin{aligned} \overline{\lim}_{h \rightarrow 0} \|Q_h(f) - P_h(f)\| &= \overline{\lim}_{h \rightarrow 0} \left\| Q_h \left( \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \right) - P_h \left( \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \right) \right\| = \\ &= \overline{\lim}_{h \rightarrow 0} \left\| \left( \sum_{i=1}^n \alpha_i Q_h(\chi_{\Delta_i}) \right) - \left( \sum_{i=1}^n \alpha_i P_h(\chi_{\Delta_i}) \right) \right\| = \\ &= \overline{\lim}_{h \rightarrow 0} \left\| \sum_{i=1}^n \alpha_i (Q_h(\chi_{\Delta_i}) - P_h(\chi_{\Delta_i})) \right\| \leq \sum_{i=1}^n |\alpha_i| \lim_{h \rightarrow 0} \|Q_h(\chi_{\Delta_i}) - P_h(\chi_{\Delta_i})\| = 0. \end{aligned}$$

therefore

$$\lim_{h \rightarrow 0} \|Q_h(f) - P_h(f)\| = 0, \quad \forall f \in S_0(X).$$

Let  $f \in B_0(X)$ . Then there are functions  $(f_n)_{n \in \mathbb{N}} \subset S_0(X)$  such that  $f \rightarrow f_n$ . Since  $Q_h, P_h \in B(H)$  and from above relation, we have

$$\lim_{h \rightarrow 0} \|Q_h(f) - P_h(f)\| = 0, \forall f \in B_0(X).$$

Reciprocal. We suppose that  $\{Q_h\}, \{P_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  are asymptotically equivalent, i.e.  $\lim_{h \rightarrow 0} \|Q_h(f) - P_h(f)\| = 0, \forall f \in B_0(X)$ .

Taking  $f = \chi_\Delta$  we have

$$\lim_{h \rightarrow 0} \|A_h(\Delta) - B_h(\Delta)\| = \lim_{h \rightarrow 0} \|Q_h(\chi_\Delta) - P_h(\chi_\Delta)\| = 0, \forall \Delta \in \Sigma_X.$$

□

**Proposition 33.** *Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure having property  $A_h(C_X) \subset B \forall h \in (0,1]$  and  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  the corresponding positive asymptotic morphism. Then*

$$\text{spec}(A_h) = \text{supp}(Q_h), \forall h \in (0,1].$$

*Proof.* Let  $\Delta$  be an open set such that  $\text{supp}(Q_h) \cap \Delta = \emptyset$ . Then

$$Q_h(f) = 0, \forall f \text{ with } \text{supp}(f) \subset \Delta.$$

Taking  $f = \chi_\Delta$ , we have  $A_h(\Delta) = Q_h(\chi_\Delta) = 0$ . Thus  $\Delta \subset X \setminus \text{spec}(A_h)$ , for each open set  $\Delta$  such that  $\text{supp}(Q_h) \cap \Delta = \emptyset$ . Therefore

$$\text{spec}(A_h) \subseteq \text{supp}(Q_h), \forall h \in (0,1].$$

Reciprocal. Let  $\Delta$  be an open set such that  $\Delta \subset X \setminus \text{spec}(A_h)$ . Then  $Q_h(\chi_\Delta) = A_h(\Delta) = 0$ .

Let  $f \in S_0(X)$  such that  $\text{supp}(f) \subset X \setminus \text{spec}(A_h)$ . Then there are disjoint sets  $(\Delta_i)_{i=1,n}$  such that  $f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}$  and  $\Delta_i \subset X \setminus \text{spec}(A_h)$ ,  $\forall i = \overline{1,n}$ . From the preceding relation, we have

$$Q_h(f) = Q_h\left(\sum_{i=1}^n \alpha_i \chi_{\Delta_i}\right) = \sum_{i=1}^n \alpha_i Q_h(\chi_{\Delta_i}) = 0.$$

Let  $f \in B_0(X)$  such that  $\text{supp}(f) \subset X \setminus \text{spec}(A_h)$ . Then there are functions  $(f_n)_{n \in \mathbb{N}} \subset B_0(X)$  such that  $f \rightarrow f_n$  and  $\text{supp}(f_n) \subset X \setminus \text{spec}(A_h)$ ,  $\forall n \in \mathbb{N}$ . From the preceding relation, we have  $Q_h(f) = 0, \forall f \in B_0(X)$  such that  $\text{supp}(f) \subset X \setminus \text{spec}(A_h)$ . Thus

$$\text{supp}(Q_h) \subseteq \text{spec}(A_h), \forall h \in (0,1].$$

□



**Remark 34.** Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure having property  $A_h(C_X) \subset B \forall h \in (0,1]$  and  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  the corresponding positive asymptotic morphism. Then  $\{A_h\}$  has compact support if and only if  $\{Q_h\}$  has compact support.

**Proposition 35.** Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure having compact support and  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  the corresponding positive asymptotic morphism. The corresponding positive asymptotic morphism of asymptotic spectral measure  $\{A_h^a\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$ , given by  $A_h^a(b) = A_h(a \cap b) \forall b \in B$  and  $\forall h \in (0,1]$ , is  $\{Q_h^a\}_{h \in (0,1]} : B_0 \rightarrow B(H)$  given by

$$Q_h^a(f) = Q_h(\chi_a f), \quad \forall f \in B_0(X) \text{ and } \forall h \in (0,1].$$

*Proof.* Since  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  is an asymptotic spectral measure having compact support, by Proposition 23 follows that  $\{A_h^a\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$ , given by relation  $A_h^a(b) = A_h(a \cap b)$  for  $\forall b \in B$  and  $\forall h \in (0,1]$ , is an asymptotic spectral measure. In addition, by Remark of Proposition 24, results that  $\{A_h^a\}$  has a compact support.

Taking into account the following relation  $\chi_{a \cap b} = \chi_a \chi_b$ , we have

$$A_h^a(b) = A_h(a \cap b) = Q_h(\chi_{a \cap b}) = Q_h(\chi_a \chi_b) = Q_h^a(\chi_b),$$

$\forall b \in B$  and  $\forall h \in (0,1]$ .

Let  $f \in B_0(\mathbb{C})$ . Since  $\{Q_h\}$  is the corresponding positive asymptotic morphism of asymptotic spectral measure  $\{A_h\}$  and having in view that

$$A_h^a(x) = \chi_a(x) A_h(x),$$

we have that

$$\begin{aligned} Q_h^a(f) &= \int_X f(x) \chi_a(x) dA_h(x) = \\ &= \int_a f(x) dA_h(x) = \int_X f(x) d\chi_a(x) A_h(x) = \int_X f(x) dA_h^a(x). \end{aligned}$$

□

**Remark 36.** Even if  $\{Q_h\}$  is a positive asymptotic morphism,  $\{Q_h^a\}$  is not necessary a positive asymptotic morphism, because  $Q_h^a(1) = Q_h(\chi_a)$ ,  $\forall h \in (0,1]$ .

**Corollary 37.** Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  and  $\{A_h^a\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  as in the preceding Proposition. Then

$$\lim_{h \rightarrow 0} \left\| \int_X f(x) dA_h^a(x) - A_h(a) \int_X f(x) dA_h(x) \right\| = 0.$$

$\forall f \in B_0(X)$ .

*Proof.* Let  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  the corresponding positive asymptotic morphism of  $\{A_h\}$  and  $\{Q_h^a\}$  given by  $Q_h^a(f) = Q_h(\chi_a f)$ ,  $\forall f \in B_0(X)$ . Then

$$\begin{aligned} \overline{\lim}_{h \rightarrow 0} \left\| \int_X f(x) dA_h^a(x) - A_h(a) \int_X f(x) dA_h(x) \right\| &= \overline{\lim}_{h \rightarrow 0} \|Q_h^a(f) - A_h(a)Q_h(f)\| = \\ &= \overline{\lim}_{h \rightarrow 0} \|Q_h(\chi_a f) - Q_h(\chi_a)Q_h(f)\| = 0. \end{aligned}$$

$\forall f \in B_0(X)$ . □

**Corollary 38.** *Let  $\{Q_h\}, \{P_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  be two positive asymptotic morphisms and  $\{Q_h^a\}, \{P_h^a\}_{h \in (0,1]} : B_0 \rightarrow B(H)$  as in Proposition 5.5. Then  $\{Q_h^a\}, \{P_h^a\}_{h \in (0,1]}$  are asymptotically equivalent if and only if  $\{Q_h\}, \{P_h\}_{h \in (0,1]}$  are asymptotically equivalent for any  $a \in B$ .*

*Proof.* Applying Proposition 35 and Proposition 26 □

**Theorem 39.** *Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure having property  $A_h(C_X) \subset \mathcal{B} \forall h \in (0,1]$  and  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  the corresponding positive asymptotic morphism. Then*

$$\text{spec}(\{A_h\}) = \text{supp}(\{Q_h\}).$$

*Proof.* Let  $\Delta$  be an open set such that  $\text{supp}(\{Q_h\}) \cap \Delta = \emptyset$ . Then

$$\lim_{h \rightarrow 0} \|Q_h(f)\| = 0, \forall f \text{ with } \text{supp}(f) \subset \Delta.$$

Taking  $\chi_\Delta$ , we have

$$\lim_{h \rightarrow 0} \|A_h(\Delta)\| = \lim_{h \rightarrow 0} \|Q_h(\chi_\Delta)\| = 0.$$

Thus  $\Delta \subset X \setminus \text{spec}(\{A_h\})$ , for each open set  $\Delta$  such that  $\text{supp}(\{Q_h\}) \cap \Delta = \emptyset$ . Therefore

$$\text{spec}(\{A_h\}) \subseteq \text{supp}(\{Q_h\}).$$

Reciprocal. Let  $\Delta$  be an open set such that  $\Delta \subset X \setminus \text{spec}(A_h)$ . Then

$$Q_h(\chi_\Delta) = A_h(\Delta) = 0.$$

Let  $f \in S_0(X)$  such that  $\text{supp}(f) \subset X \setminus \text{spec}(\{A_h\})$ . Then there are disjoint sets  $(\Delta_i)_{i=1,n}$  such that  $f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}$  and  $\Delta_i \subset X \setminus \text{spec}(\{A_h\})$ ,  $\forall i = \overline{1,n}$ . From the previous relation, we have

$$Q_h(f) = Q_h\left(\sum_{i=1}^n \alpha_i \chi_{\Delta_i}\right) = \sum_{i=1}^n \alpha_i Q_h(\chi_{\Delta_i}) = 0.$$

When  $h \rightarrow 0$  into the prior relation, it results

$$\lim_{h \rightarrow 0} \|Q_h(f)\| = 0,$$

$\forall f \in S_0(X)$  such that  $\text{supp}(f) \subset X \setminus \text{spec}(\{A_h\})$ .

Let  $f \in B_0(X)$  such that  $\text{supp}(f) \subset X \setminus \text{spec}(A_h)$ . Then there are functions  $(f_n)_{n \in \mathbb{N}} \subset S_0(X)$  such that  $f \rightarrow f_n$  and  $\text{supp}(f_n) \subset X \setminus \text{spec}(A_h)$ ,  $\forall n \in \mathbb{N}$ . From the previous relation, we have

$$\lim_{h \rightarrow 0} \|Q_h(f)\| = 0,$$

$\forall f \in B_0(X)$  such that  $\text{supp}(f) \subset X \setminus \text{spec}(A_h)$ .

Therefore

$$\text{supp}(\{Q_h\}) \subseteq \text{spec}(\{A_h\}).$$

□

**Corollary 40.** Let  $\{Q_h\}, \{P_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  be two corresponding positive asymptotic morphisms having property  $Q_h(C_0(X)) \subset \mathcal{B}$  and  $P_h(C_0(X)) \subset \mathcal{B} \forall h \in (0,1]$ . Then

$$\text{supp}(\{Q_h\}) = \text{supp}(\{P_h\}).$$

*Proof.* By Theorem 39 and Theorem 21. □

**Corollary 41.** Let  $\{Q_h\}, \{P_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  be two corresponding positive asymptotic morphisms of two compact asymptotic spectral measures. Then  $\{Q_h\}$  has compact support if and only if  $\{P_h\}$  has compact support.

*Proof.* By Theorem 39 and Theorem 21. □

**Theorem 42.** Let  $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$  be an asymptotic spectral measure having compact support and  $\{A_h^a\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$ , given by  $A_h^a(b) = A_h(a \cap b) \forall b \in B$  and  $\forall h \in (0,1]$ . Then

$$\text{spec}(\{A_h^a\}) = \overline{a} \cap \text{spec}(\{A_h\}), \forall h \in (0,1].$$

*Proof.* We show that

$$\operatorname{spec}(\{A_h^a\}) \subseteq \overline{a} \cap \operatorname{spec}(\{A_h\}).$$

By Proposition 24 we have

$$\operatorname{spec}(A_h^a) \subset \overline{a} \cap \operatorname{spec}(A_h).$$

By Remark 18 i), it follows

$$\operatorname{spec}(\{A_h^a\}) \subset \overline{a}.$$

Let  $b$  be a compact set such that  $b \subset \mathbb{C} \setminus \operatorname{spec}(\{A_h\})$ . Then there are open sets  $(b_i)_{i=1, \dots, n}$  such that  $b \subset \bigcup_{i=1}^n b_i$ ,  $b_i \subset \mathbb{C} \setminus \operatorname{spec}(\{A_h\})$  and it results

$$\lim_{h \rightarrow 0} \|A_h(b_i)\| = 0.$$

Since each  $A_h$  is additive, we have

$$A_h(b) \leq A_h\left(\bigcup_{i=1}^n b_i\right) = \sum_{i=1}^n A_h(b_i).$$

When  $h \rightarrow 0$ , it follows

$$\lim_{h \rightarrow 0} \|A_h(b)\| \leq \sum_{i=1}^n \lim_{h \rightarrow 0} \|A_h(b_i)\| = 0.$$

How

$$\begin{aligned} \lim_{h \rightarrow 0} \|A_h^a(b)\| &= \lim_{h \rightarrow 0} \|A_h(a \cap b)\| \leq \\ &\leq \sum_{i=1}^n \lim_{h \rightarrow 0} \|A_h(a \cap b_i)\| = \sum_{i=1}^n \lim_{h \rightarrow 0} \|A_h(a)A_h(b_i)\| \leq \\ &\leq \sum_{i=1}^n \lim_{h \rightarrow 0} \|A_h(b_i)\| = 0, \end{aligned}$$

it results  $b \subset \mathbb{C} \setminus \operatorname{spec}(\{A_h^a\})$  for each compact set  $b$  such that  $b \subset \mathbb{C} \setminus \operatorname{spec}(\{A_h\})$ . Therefore

$$\operatorname{spec}(\{A_h^a\}) \subseteq \operatorname{spec}(\{A_h\})$$

(by regularity of measures  $A_h$ ).

Reciprocal. Let  $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$  be the corresponding positive asymptotic morphism of  $\{A_h\}$ . We show

$$\text{supp}(\{Q_h^a\}) \supseteq \bar{a} \cap \text{supp}(\{Q_h\}),$$

where  $\{Q_h^a\}_{h \in (0,1]} : B_0 \rightarrow B(H)$  is given by

$$Q_h^a(f) = Q_h(\chi_a f).$$

Let  $a$  be a set such that  $\bar{a} \cap \text{supp}(\{Q_h\}) \subset F$ . Let  $f \in B_0(X)$  such that  $\text{supp}(f) \subset X \setminus F$ . Then  $\text{supp}(f) \cap F = \emptyset$  and thus  $\lim_{h \rightarrow 0} \|Q_h(f)\| = 0$ .

Let  $\{Q_h^a\}_{h \in (0,1]} : B_0 \rightarrow B(H)$  given by

$$Q_h^a(f) = Q_h(\chi_a f), \forall f \in B_0(X) \text{ and } \forall h \in (0, 1].$$

The, from the previous relation and since  $\lim_{h \rightarrow 0} \|Q_h(\chi_a)\| \leq 1$ , we have

$$\begin{aligned} \lim_{h \rightarrow 0} \|Q_h^a(f)\| &= \lim_{h \rightarrow 0} \|Q_h(\chi_a f)\| \leq \lim_{h \rightarrow 0} \|Q_h(\chi_a) Q_h(f)\| \leq \\ &\leq \lim_{h \rightarrow 0} \|Q_h(\chi_a)\| \lim_{h \rightarrow 0} \|Q_h(f)\| \leq \lim_{h \rightarrow 0} \|Q_h(f)\| = 0, \end{aligned}$$

$f \forall \in B_0(X)$  such that  $\text{supp}(f) \subset X \setminus F$ . Thus  $\text{supp}(\{Q_h^a\}) \subset F$ .

By Theorem 39 we have

$$\text{spec}(\{A_h\}) = \text{supp}(\{Q_h\}),$$

thus

$$\bar{a} \cap \text{spec}(\{A_h\}) \subseteq \text{spec}(\{A_h^a\}).$$

□

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